28-jan Lecture 1

Tuesday, January 27, 2026 6:53 PM

recall: implicit differentiation

given $x^2+y^2=r^2$ differentiate WRT time (t)

$$x\frac{dx}{dt}+y\frac{dy}{dt}=r\frac{dr}{dt}$$

differential equations (DE) $:=$ ← is defined as equations relating to UNK functions <u>and</u> 1+ of its derivative(s)

ex. $\frac{dx}{dt}=x^2+y^2$ ← 1st order DE since highest derivative included is <u>1</u>

UNK f is $x(t)$

$x'(t)$

ex. 2nd order DE $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+7y=0$ ← UNK y, y', y''

recall notation

given $x^2+3x-4=0$ ← find <u>numbers</u> that satisfy eq'n

to solve DE, find UNK <u>function</u> for which $y=y(x)$ is an identity

ex. show $y(x)=Ce^{x^2}$ (where C is a constant) is a solution to $\frac{dy}{dx}=2xy$

$$\frac{d}{dx}(y(x))=\frac{d}{dx}\left(Ce^{x^2}\right) \qquad 2x\left(Ce^{x^2}\right)$$

$$=C\,2xe^{x^2} \qquad =C\,2xe^{x^2} \qquad \square$$

conclude: every function $y(x)$ in the form $y=Ce^{x^2}$ is a solution to the DE $\frac{dy}{dx}=2xy$

DE doesn't necessarily have a sol'n

ex. $(y')^2+y^2+1=0 \longrightarrow (y')^2+y^2=-1$

$\quad -1 \quad -1$

sum of squares can't be equiv to

$-1 \quad -1$

sum of squares can't be equiv to a negative number

ex. $(y')^2 + y^2 = 0 \longrightarrow$ has exactly one solution: $y(x) = 0$

Parameters

ex. given A, B are constants in $y(x) = A\cos 3x + B\sin 3x$

then $y'(x) = -3A\sin 3x + 3B\cos 3x$

$y''(x) = -9A\cos 3x - 9B\sin 3x$

$y'' = -9(A\cos 3x + B\sin 3x)$

DE: $\left[\begin{array}{l} y'' = -9y \quad \text{or} \\ y'' + 9y = 0 \end{array}\right.$

DE has a <u>two</u>-parameter family of solutions on $\mathbb{R}$'s

dep. A B values

3 sol'ns of $y'' + 9y = 0$ could $y_1(x) = 3\cos 3x \quad A = 3 \quad B = 0$

$y_2(x) = 2\sin 3x \quad A = 0 \quad B = 2$

$y_3(x) = -3\cos 3x + 2\sin 3x \quad A = -3 \quad B = 2$

revisit $\frac{dy}{dx} = 2xy$ w family of solns $y = Ce^{x^2}$

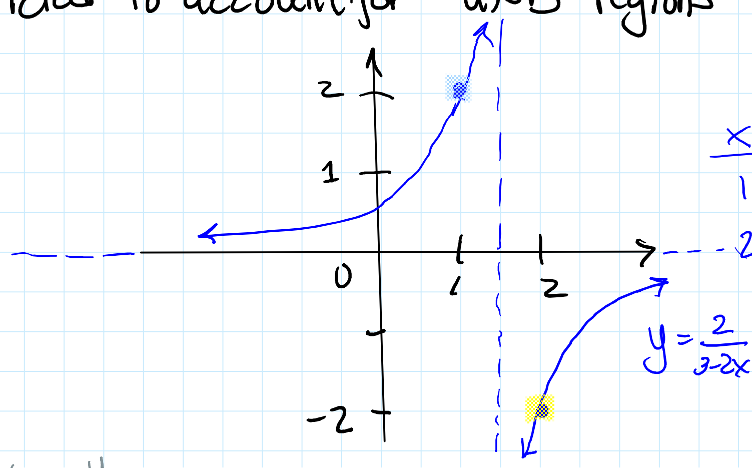
one parameter family of sol'ns

Restrictions

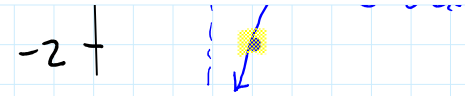
solution curves may be restricted to account for UND "regions"

graph of $y(x) = \frac{2}{3-2x}$

VA @ $x = 3/2$



x	y
1	2
2	-2



given initial condition $y(\overset{x}{1}) = \overset{y}{2}$
↑ is on LH branch of graph
$\therefore$ solution is on $(-\infty, \frac{3}{2})$

given $y(2) = -2 \Rightarrow$ solution is on curve restricted by $(\frac{3}{2}, \infty)$

ex. solve $\frac{dy}{dx} = 2x+3$ given $y(1)=2$

RHS is <u>not</u> dep. on y

Integrate WRT x: $\int \frac{dy}{dx}\,dx = \int (2x+3)\,dx$

$$y(x) = x^2 + 3x + C \quad \text{general sol'n}$$

$$y(1) = 2 = 1 + 3 + C \Rightarrow C = -2$$

$\therefore$ particular sol'n is $\boxed{y(x) = x^2 + 3x - 2}$

<u>Solving 2nd order DE:</u>

given $\frac{d^2y}{dx^2} = g(x)$ find $y(x)$

then $\frac{dy}{dx} = \int y''(x)\,dx = \int g(x)\,dx$ antiderivative

$$y'(x) = G(x) + C_1$$ where C_1 is an arbitrary constant

now $y(x) = \int y'(x)\,dx = \int (G(x) + C_1)\,dx$

$$\boxed{y(x) = \int G(x)\,dx + C_1 x + C_2}$$

also

<u>Slope Fields and Solution Curves</u>

consider $\frac{dy}{dx} = f(x, y)$

indep. variable

dependent variable

you might think its sol'n is $y(x) = \int f(x,y)\,dx + C$

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$f(x, y(x))$

problem: integral includes the UNK $y(x)$ in 'solution'

$\therefore$ cannot be evaluated explicitly

ex. $y' = x^2 + y^2$ has sol'ns that can't be easily expressed

instead, can think of its solutions geometrically to $y' = f(x,y)$ at each (x,y) on xy-plane

m

value of $f(x,y)$ is its $m \Rightarrow m = f(x,y)$

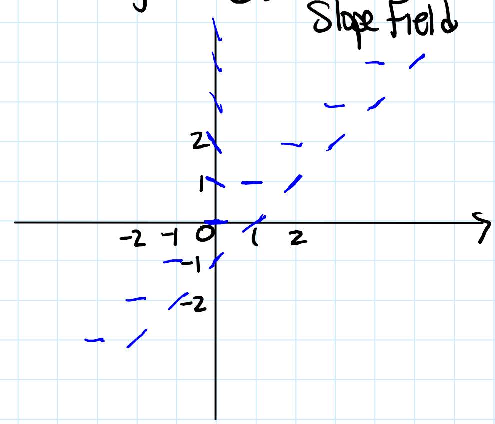
i.e., DE sol'n is a differentiable function whose graph $y = y(x)$ has a correct slope at each $(x, y(x))$ through which it passes

thus, a solution curve of DE $y' = f(x,y)$ is a curve on xy-plane whose tangent line at each (x,y) has $m = f(x,y)$

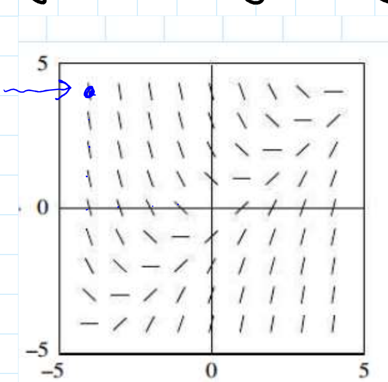
ex. given DE $y' = x - y$

x	y	$y' = m$
0	0	0
0	2	-2
1	0	1
0	1	-1
1	1	0
-1	0	-1

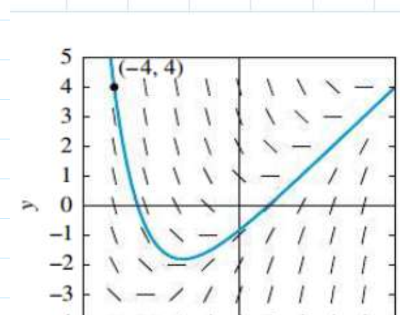
Slope Field



(-4,4)



then the particular solution through $(-4,4)$



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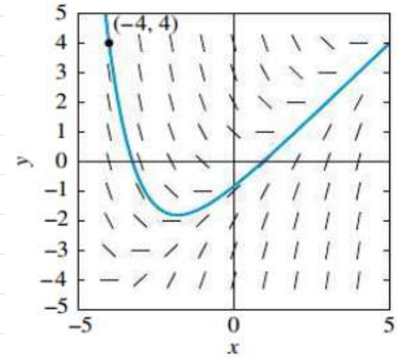
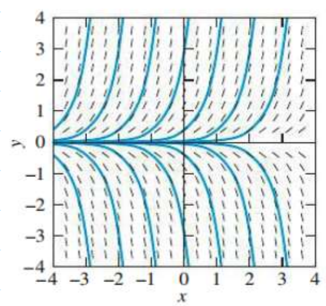


ex. given $\frac{dy}{dx} = ky$ look at slope field and sol'n curve when $k = 2$

$y' = 2y$

y	y'
0	0
1	2
2	4
-1	-2



better figure than what was originally pasted